Intermediate test Waves and Optics - 7 December 2015 – 9:00-11:00

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This test contains 4 questions on xxx pages.

A few **preliminary remarks**:

- Please answer questions 3 & 4 on another (double) sheet of paper than questions 1 & 2.
- Put your name and student number at the top of all sheets.
- Put your student card at the edge of the desk for checking by the assistants and show it when handing in your test.
- Add the units to the numbers calculated.

Question 1 (6 points): plane harmonic waves

A plane harmonic electromagnetic wave is specified (in SI units) by the following wave function:

$$\vec{\mathbf{E}} = \left(4\hat{\mathbf{i}} - 6\hat{\mathbf{j}}\right)\left(10^3\ \mathrm{V/m}\right)\ \cos\left[\left(3x + 2y\right)\pi \times 10^7 - 1.26 \times 10^{16} t\right]$$

with $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ the unit vectors along the *x*- and *y*-axis, respectively.

Questions:

a) Draw the direction in the *x-y* plane along which the electric field oscillates.

b) What is the scalar value of the amplitude of the electric field ?

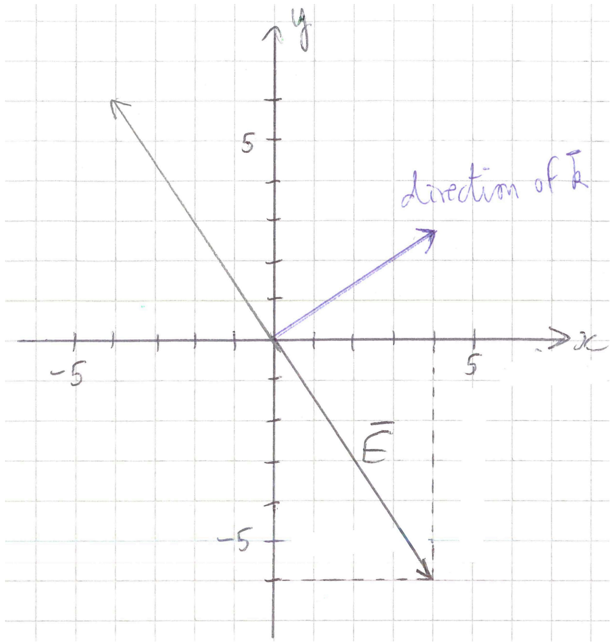
c) What is the direction of propagation of the wave ? Indicate this direction in the drawing used in the answer of question a).

d) What is the wavelength of the wave ?

e) What is the frequency of the wave ?

Answer

a)



b) $|\vec{\mathbf{E}}| = (\vec{\mathbf{E}} \bullet \vec{\mathbf{E}})^{1/2} = \left[\mathrm{E}_x^2 + \mathrm{E}_y^2 + \mathrm{E}_z^2 \right]^{1/2}$

$$= \left[4^2 + (-6)^2 + 0^2 \right]^{1/2} 10^3 \text{ V/ m} = 7.21 \times 10^3 \text{ V/ m}$$

c) For a harmonic plane wave, the part of the phase containing the spatial variables is equal to $\vec{k} \bullet \vec{r}$, with $\vec{k}$ the propagation vector with direction equal to the direction of propagation of the wave. For the given wave: $\vec{k} \bullet \vec{r} = k_x x + k_y y + k_z z = (3x + 2y)\pi \times 10^7$

the direction of $\vec{k}$, and thus also the direction of propagation is given by the vector $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

As the wave function depends on "$\vec{k} \bullet \vec{r} - \omega t$" (minus sign !), the wave propagates in the direction indicated by the arrow in the figure.

d) wavelength $\lambda = \dfrac{2\pi}{k}$

$$k = |\vec{k}| = (\vec{k} \bullet \vec{k})^{1/2} = \pi \times 10^7 \left[3^2 + 2^2 \right]^{1/2} = 3.61\pi \times 10^7 \text{ / m}$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{3.61\pi \times 10^7} = 0.554 \times 10^{-7} \text{ m} = 55.4 \text{ nm}$$

e) from the wave function: $\omega = 1.26 \times 10^{16}$ rad/ s frequency $\nu = \frac{\omega}{2\pi} = \frac{1.26\times10^{16}}{2\pi} = 1.91\times10^{15}\,/\,\mathrm{s}$

Question 2 (4 points): Intensity and amplitude of an electromagnetic wave

Assume a laser beam (i.e. a harmonic electromagnetic wave) in vacuum with a power of 2000 W that is concentrated to a cross section of 10^{-6} cm^2.

<u>Questions</u>:

1. What is the irradiance of this beam ?
2. What is the amplitude of the electric field of the corresponding harmonic electromagnetic wave ?
3. What is the amplitude of the magnetic field of the corresponding harmonic electromagnetic wave ?

Give the physical units for all answers !

Extra information:
The size of the Poynting vector is given by: $S(t) = c^2\, \varepsilon_0\, E(t)\, B(t)$
with: c the speed of light in vacuum = 3×10^8 m/s
ε_0 the permittivity of the vacuum = 8.854×10^{-12} J/(V^2 m)
$E(t)$, $B(t)$ the size of the electric, magnetic field

Answer:

1. irradiance ≡ time average of the energy per unit of time and unit of surface area = power per unit of surface area
   irradiance = 2000 W / 10^{-6} cm^2 = 2×10^9 W/cm^2 (= 2×10^{13} W/m^2)

for a harmonic electromagnetic wave (moving in e.g. the x-direction):

$$I \equiv \langle S(t)\rangle_T = c^2\varepsilon_0 E_0 B_0 \left\langle \cos^2(kx-\omega t)\right\rangle_T = \tfrac{1}{2}c^2\varepsilon_0 E_0 B_0$$

from $B_0 = \frac{1}{c}E_0$ follows $I = \tfrac{1}{2}c\varepsilon_0 E_0^2$

2. $$E_0 = \sqrt{\frac{2I}{c\varepsilon_0}} = \sqrt{\frac{2\times2\times10^{13}\ W/m^2}{3\times10^8\times8.854\times10^{-12}(m/s)(J/V^2/m)}}$$

$$= 1.23\times10^8\sqrt{\frac{J\ s\ V^2\ m}{s\ m^2\ m\ J}} = 1.23\times10^8\sqrt{\frac{V^2}{m^2}} = 1.23\times10^8\,\frac{V}{m}$$

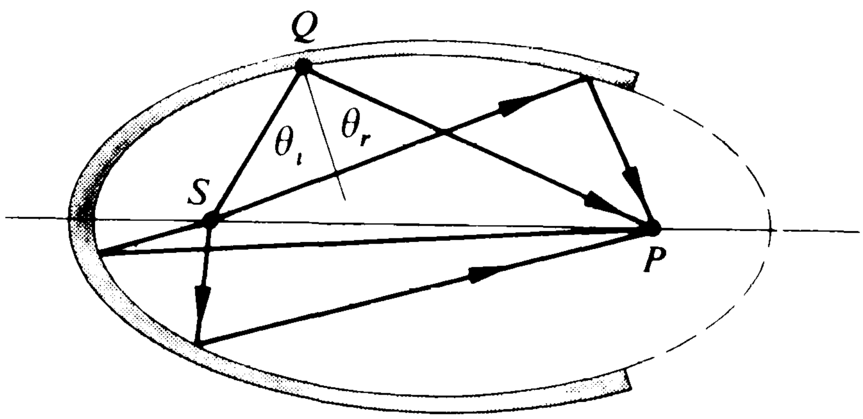
3. $$B_0 = \frac{1}{c}E_0 = \frac{1.23\times10^8}{3\times10^8}\frac{V\ s}{m^2} = 0.41\ T$$

Question 3 (6 points): Fermat's principle

Fermat's principle allows to determine the manner in which light propagates.

a) What does the **original formulation** of Fermat's principle say about the path light will follow ? Give **two** equivalent versions. (No derivations are asked for, just state the principle.)

b) In some situations, Fermat's principle is not valid. In the lectures, an example of such a situation was discussed involving a source *S* and observation point *P* in the focii of an ellipsoid. The figure below should bring this situation to mind. For an ellipsoid, all paths SQP (with Q any point on the ellipsoid) have the same length.



Use this example to explain a situation in which Fermat's principle is not valid.

c) The existance of situations in which Fermat's principle is not valid has resulted in a **modern formulation** of Fermat's principle which is always valid. Give this modern formulation and explain its meaning.

Answer:

a)

Original formulation:
1) Light follows the fastest path, shortest time.
2) Light follows the path with the smallest optical path length.

b)

Consider a curved mirror inside the ellipsoid with the same tangent in point *Q* as the ellipsoid. This does not change anything to the path of the ray *SQP*: light going form *S* to *P* will still reflect at *Q*. Rays reflected off the curved mirror next to the point *Q* will now have a shorter optical path and thus a shorter time to go from *S* to *P*. Note that since the rays propagate in one medium only, the shortest optical path is also the shortest path. So the path *SQP* which the light follows has the longest optical path length and is thus the slowest.

c)
Light follows a path such that the optical path length is stationary with respect to variations of that path. A stationary path is a path for which the optical path length of neighbouring paths is not very different.

Question 4 (4 points): Standing waves

The following expression represents a standing wave:

$$E = 150\ \sin\tfrac{1}{3}\pi x\ \ \cos 4\pi t$$

(x is the spatial coordinate, t is time)

Questions:

a) Why is this called a "standing" wave ?
b) Give 2 wave functions which, when superposed, generate the standing wave given above.

Answer:

a) This is called a standing wave because the wave profile does not travel through space but is standing still.
b) A standing wave is the superposition of 2 "traveling" waves traveling in opposite directions, having the same frequency and amplitude. The amplitude of the resulting standing wave is twice the amplitude of the individual waves. So for the standing wave given above:

$$E = E_1 + E_2\ with$$

$$E_1 = 75\ \sin\left(\tfrac{1}{3}\pi x + 4\pi t\right)$$

$$E_2 = 75\ \sin\left(\tfrac{1}{3}\pi x - 4\pi t\right)$$